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Formulation of Equations of Motion for Complex Spacecraft

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I. Introduction

THE formulation of equations of motion, always an important facet of spacecraft design and performance analysis, is receiving ever greater attention as vehicles become larger and more complex and as the need grows more and more acute to predict spacecraft motions accurately. This article is intended to answer the following question that is clearly central in this context: "What are the relative advantages of various methods available for the formulation of equations of motion of complex spacecraft?"

The question just stated can be answered incisively only if the criteria by which a given method is judged are sufficiently well defined, and meaningful criteria can be established only after a clear-cut purpose for the formulation of equations of motion has been identified. We take this purpose to be the production of algorithms to be employed for the simulation of spacecraft motions by means of numerical solutions of initial value problems. The principal criteria by which we judge a method are the *simplicity* of the equations to which it leads and the amount of *labor* required for the formulation of the equations. Of course, neither criterion is important when one is dealing with a relatively uncomplicated system. Conversely, these matters can become crucial in connection with truly complex spacecraft, for both computer storage limitations and excessive execution times can then plague one needlessly if equations have not been brought into the simplest possible form, and the labor expended to generate equations which have such a form can easily become prohibitive unless a highly efficient scheme for formulating equations of motion has been employed from the outset. It should be understood, therefore, that we are not concerned with systems so simple that their equations of motion can be produced with essentially equal ease by using almost any method.

Since there exist numerous so-called multibody programs, as well as analyses intended to be the bases for such programs, and it is quite clear that these are of great value in solving problems of spacecraft dynamics, one may well ask whether or not it is ever really necessary to construct literal equations of motion for complex vehicles. The answer is that the need to do this arises frequently because multibody programs can fail a user in a number of ways. Every such program represents a compromise between two mutually exclusive goals, that of providing a computer code applicable to the solution of the widest possible class of problems, and that of minimizing the effort required by a user to bring his problem into a form compatible with program input requirements. Consequently, a given multibody program can be totally inapplicable to a particular problem, can force a user to make major program additions or other modifications, or can lead to inefficient or inaccurate simulations. Clearly, therefore, algorithms created to meet specific needs are indispensable. Not surprisingly, it is precisely the authors of some of the best multibody programs who, realizing all of this, welcome algorithms developed independently, particularly since these can be used to test the validity of multibody computer codes. Moreover, although it is probably true that the greatest obstacle to be surmounted in producing a multibody program is that of devising a procedure for assembling the equations of motion associated with the various bodies comprising a spacecraft, rather than that of writing equations governing the behavior of a generic body, there can be little doubt that the way in which the latter

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task is handled can have a substantial impact on computer storage requirements and execution times. Since the method we ultimately find to be the most efficacious for the formulation of literal equations also furnishes an excellent basis for multibody programs, there exists a harmonious relationship, rather than a conflict, between our present concerns and those of the proponents of multibody programs.

At present, flexible spacecraft, that is, systems that cannot be modeled as a collection of rigid bodies, are of particular interest. What follows deals both with systems possessing a finite number of degrees of freedom and with flexible spacecraft whenever the behavior of the latter can be analyzed in terms of modal coordinates.

To support the conclusions that are to be drawn, we focus attention initially on a specific example of the kind encountered in the field of spacecraft dynamics, choosing for this a system that is sufficiently complex to permit one to recognize significant differences in the structure of governing equations and in the effort involved in their formulation by various methods, but that is at the same time sufficiently simple to allow us to report results within the confines of an article such as the present one. Hopefully, it will be clear that the arguments advanced in connection with this example apply quite generally and *a fortiori* to more complicated systems.

Seven methods are examined, namely the use of momentum principles,[1] D'Alembert's principle,[2] Lagrange's equations,[3] Hamilton's canonical equations,[4] the Boltzmann-Hamel equations,[5,6] the Gibbs equations,[7] and a method[8] introduced in 1965, in that order. The first five methods are the ones probably most familiar to the majority of workers in the field of spacecraft dynamics, and certainly used most widely; the sixth, use of the Gibbs equations, is encountered only infrequently in the modern literature, but is included both because it has considerable merit and because it furnishes a natural transition to the last method, which, as will be shown, is the one that leads most directly to the simplest equations. This method is the only one originating during the second half of the twentieth century, which may account for the fact that it is better suited than the rest for dealing with twentieth-century problems of spacecraft dynamics.

Comments regarding good and bad features of each of the seven approaches considered are made as the occasions present themselves. A reader who does not wish to follow these developments in detail can turn directly to Sec. X, where the observations made in Sec. III-IX are summarized.

II. Example

Figure 1 is a schematic representation of a spacecraft S modeled as a rigid body A that carries a planar linkage which, in turn, supports a particle P. Equal torsional springs and dampers at points L_1 and L_4 resist rotation of the linkage relative to A, and an extensional spring and damper connect P to point L_2 (dampers are not shown). The linkage is considered massless and has the dimensions indicated. P has a mass m, and A a mass M. A^* is the mass center of A; $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$ are unit vectors parallel to the central principal axes of inertia of A; the associated central principal moments of inertia are I_1, I_2, I_3; and the linkage lies in the central principal plane of A normal to $\boldsymbol{a}_3$. The resultant force exerted on P by the extensional spring and damper is given by $\sigma\boldsymbol{a}_2$, where σ is a function of r and $\dot{r}$, while the torque associated with the action of each torsional spring and damper on a link is $\tau\boldsymbol{a}_3$, with τ a function of θ and $\dot{\theta}$. Finally, external forces acting on S are a force $R_1\boldsymbol{a}_1+R_2\boldsymbol{a}_2+R_3\boldsymbol{a}_3$ applied to P and a system of forces acting on A, the latter being represented by a force $S_1\boldsymbol{a}_1+S_2\boldsymbol{a}_2+S_3\boldsymbol{a}_3$ applied at A^*, and a torque $T_1\boldsymbol{a}_1+T_2\boldsymbol{a}_2+T_3\boldsymbol{a}_3$.

Equations are to be formulated for the purpose of simulating the motion of S in a Newtonian reference frame N subsequent to an instant at which r, θ, $\dot{r}$, $\dot{\theta}$, the orientation of A in N, the angular velocity of A in N, the position of A^* in

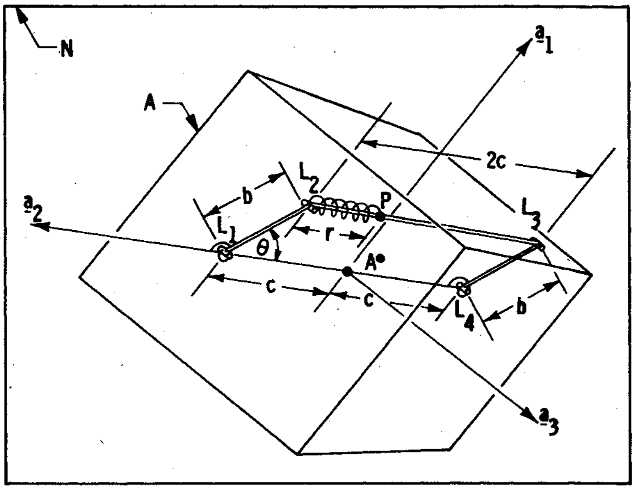

Fig. 1 Schematic representation of spacecraft.

N, and the velocity of A^* in N all are known. Clearly, therefore, one task that must be faced is that of selecting suitable scalar variables. One possibility is to use as attitude variables, denoted by q_1, q_2, q_3, the amounts of three rotations by means of which A can be brought from its initial orientation to any other, performing these rotations successively about axes parallel to $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$ after first aligning $\boldsymbol{a}_i$ with $\boldsymbol{n}_i$ $(i=1,2,3)$, where $\boldsymbol{n}_1$, $\boldsymbol{n}_2$, $\boldsymbol{n}_3$ are unit vectors fixed in N; and to describe the angular velocity $\boldsymbol{\omega}$ of A in N in terms of ω_1, ω_2, ω_3, defined as $\omega_i \triangleq \boldsymbol{\omega}\cdot\boldsymbol{a}_i$ $(i=1,2,3)$. So called rotational kinematical differential equations can then be formulated immediately, and, if s_i and c_i denote sin q_i and cos q_i $(i=1,2,3)$, respectively, these are

$$\dot{q}_1=(\omega_1 c_3-\omega_2 s_3)/c_2 \tag{1a}$$

$$\dot{q}_2=\omega_1 s_3+\omega_2 c_3 \tag{1b}$$

$$\dot{q}_3=(-\omega_1 c_3+\omega_2 s_3)s_2/c_2+\omega_3 \tag{1c}$$

Of course, attitude variables can be chosen in a variety of other ways, some of which may in fact be superior (e.g., Euler parameters, direction cosines). This is a point to which we shall return after we have disposed of the task of formulating dynamical equations. First, however, we must address the matter of position and velocity variables. As to the former, one may employ the products $\boldsymbol{p}\cdot\boldsymbol{n}_i$ $(i=1,2,3)$, where $\boldsymbol{p}$ is the position vector from a point fixed in N to A^*, calling these products, say, q_4, q_5, q_6, respectively, and, to deal with the velocity $\boldsymbol{v}^{A^*}$ of A^*, one can introduce $v_i=\boldsymbol{v}^{A^*}\cdot\boldsymbol{a}_i$ $(i=1,2,3)$. The translational kinematical differential equations to which this leads are

$$\dot{q}_4=v_1 c_2 c_3-v_2 c_2 s_3+v_3 s_2 \tag{2a}$$

$$\dot{q}_5=v_1(s_1 s_2 c_3+s_3 c_1)+v_2(-s_1 s_2 s_3+c_3 c_1)-v_3 s_1 c_2 \tag{2b}$$

$$\dot{q}_6=v_1(-c_1 s_2 c_3+s_3 s_1)+v_2(c_1 s_2 s_3+c_3 s_1)+v_3 c_1 c_2 \tag{2c}$$

III. Momentum Principles

The use of momentum principles to formulate dynamical equations requires time-differentiation of linear and angular momenta. The linear momentum of S is $M\boldsymbol{v}^{A^*}+m\boldsymbol{v}^P$, where $\boldsymbol{v}^{A^*}$ and $\boldsymbol{v}^P$, the velocities of A^* and P, are given by

$$\boldsymbol{v}^{A^*}=v_1\boldsymbol{a}_1+v_2\boldsymbol{a}_2+v_3\boldsymbol{a}_3 \tag{3}$$

and

$$v^P = [v_1 + b\dot{\theta}\cos\theta - \omega_3(c-r-b\cos\theta)]a_1 + (v_2 + b\dot{\theta}\sin\theta - \dot{r} + \omega_3 b\sin\theta)a_2 + [v_3 + \omega_1(c-r-b\cos\theta) - \omega_2 b\sin\theta]a_3 \tag{4}$$

respectively. These expressions play a role also in connection with the angular momentum of S with respect to the mass center S^* of S, which is given by $I\cdot\omega + Mr^{A^*}\times v^{A^*} + mr^P \times v^P$, where I is the central inertia dyadic of A, ω is given by

$$\omega = \omega_1 a_1 + \omega_2 a_2 + \omega_3 a_3 \tag{5}$$

and r^{A^*} and r^P are the position vectors from S^* to A^* and to P, respectively. Equating the time derivative in N of the linear momentum to the resultant of the external forces, $(R_1+S_1)a_1 + (R_2+S_2)a_2(R_3+S_3)a_3$, and that of the angular momentum to the sum of the moments about S^* of the external forces, one obtains three "translational" dynamical differential equations,

$$M(\dot{v}_1 + \omega_2 v_3 - \omega_3 v_2) + m[\dot{v}_1 + \omega_2 v_3 - \omega_3 v_2 + b\ddot{\theta}\cos\theta - (\dot{\omega}_3 - \omega_1\omega_2)(c-r-b\cos\theta) - 2\omega_3(b\dot{\theta}\sin\theta - \dot{r})$$
$$- (\omega_2^2 + \omega_3^2)b\sin\theta - b\dot{\theta}^2\sin\theta] = R_1 + S_1 \tag{6}$$

$$M(\dot{v}_2 + \omega_3 v_1 - \omega_1 v_3) + m[\dot{v}_2 + \omega_3 v_1 - \omega_1 v_3 + b\ddot{\theta}\sin\theta + b\dot{\theta}^2\cos\theta - \ddot{r} + (\dot{\omega}_3 + \omega_1\omega_2)b\sin\theta + 2\omega_3 b\dot{\theta}\cos\theta$$
$$- (\omega_3^2 + \omega_1^2)(c-r-b\cos\theta)] = R_2 + S_2 \tag{7}$$

$$M(\dot{v}_3 + \omega_1 v_2 - \omega_2 v_1) + m[\dot{v}_3 + \omega_1 v_2 - \omega_2 v_1 + (\dot{\omega}_1 + \omega_2\omega_3)(c-r-b\cos\theta) - (\dot{\omega}_2 - \omega_3\omega_1)b\sin\theta$$
$$+ 2\omega_1(b\dot{\theta}\sin\theta - \dot{r}) - 2\omega_2 b\dot{\theta}\cos\theta] = R_3 + S_3 \tag{8}$$

and, with μ in place of $mM/(m+M)$, three "rotational" dynamical differential equations, the first of which is

$$I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 + \mu\Big\langle(b\dot{\theta}\sin\theta - \dot{r})[\omega_1(c-r-b\cos\theta) - \omega_2 b\sin\theta] + (c-r-b\cos\theta)[\dot{\omega}_1(c-r-b\cos\theta)$$
$$+ \omega_1(b\dot{\theta}\sin\theta - \dot{r}) - \dot{\omega}_2 b\sin\theta - \omega_2\dot{\theta}b\cos\theta] + \omega_2[b^2\dot{\theta} + b\sin\theta(\omega_3 b\sin\theta - \dot{r}) - (c-r)b\dot{\theta}\cos\theta$$
$$+ (c-r-b\cos\theta)^2\omega_3] + \omega_3 b\sin\theta[\omega_1(c-r-b\cos\theta) - \omega_2 b\sin\theta]\Big\rangle = T_1 + \mu(c-r-b\cos\theta)(R_3/m - S_3/M) \tag{9}$$

With a considerable amount of labor, one can reduce this to

$$I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 + \mu(c-r-b\cos\theta)[(\dot{\omega}_1 + \omega_2\omega_3)(c-r-b\cos\theta) - (\dot{\omega}_2 - \omega_3\omega_1)b\sin\theta + 2b\dot{\theta}(\omega_1\sin\theta - \omega_2\cos\theta) - 2\dot{r}\omega_1]$$
$$= T_1 + \mu(c-r-b\cos\theta)(R_3/m - S_3/M) \tag{10}$$

The remaining two equations, simplified similarly, are

$$I_2\dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1 - \mu b\sin\theta[(\dot{\omega}_1 + \omega_2\omega_3)(c-r-b\cos\theta) - (\dot{\omega}_2 - \omega_3\omega_1)b\sin\theta + 2b\dot{\theta}(\omega_1\sin\theta - \omega_2\cos\theta) - 2\dot{r}\omega_1]$$
$$= T_2 - \mu b\sin\theta(R_3/m - S_3/M) \tag{11}$$

and

$$I_3\dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2 + \mu\{b\sin\theta[(\ddot{\theta} + \dot{\omega}_3 + \omega_1\omega_2)b\sin\theta + (\dot{\theta} + 2\omega_3)b\dot{\theta}\cos\theta - \ddot{r} - \omega_1^2(c-r-b\cos\theta)]$$
$$- (c-r-b\cos\theta)[b\ddot{\theta}\cos\theta - (\dot{\omega}_3 - \omega_1\omega_2)(c-r-b\cos\theta) - 2\omega_3(b\dot{\theta}\sin\theta - \dot{r}) - (\dot{\theta}^2 + \omega_2^2)b\sin\theta]\}$$
$$= T_3 + \mu b\sin\theta(R_2/m - S_2/M) - \mu(c-r-b\cos\theta)(R_1/m - S_1/M) \tag{12}$$

Two additional dynamical differential equations are required, for S possesses eight degrees of freedom in N. To generate these equations, one may begin by drawing free-body diagrams of the link $L_1 - L_2$, the link $L_2 - L_3$ together with the particle P, and the link $L_3 - L_4$, as shown in Figs. 2a-2c, where $D_1,...,D_8$ denote reaction forces. Next, applying the angular momentum principle in connection with Figs. 2a, 2b, and 2c, one can write

$$b(D_3\sin\theta - D_4\cos\theta) = \tau \tag{13}$$

$$rD_4 + (2c-r)D_6 = 0 \tag{14}$$

and

$$b(D_6\cos\theta - D_5\sin\theta) = \tau \tag{15}$$

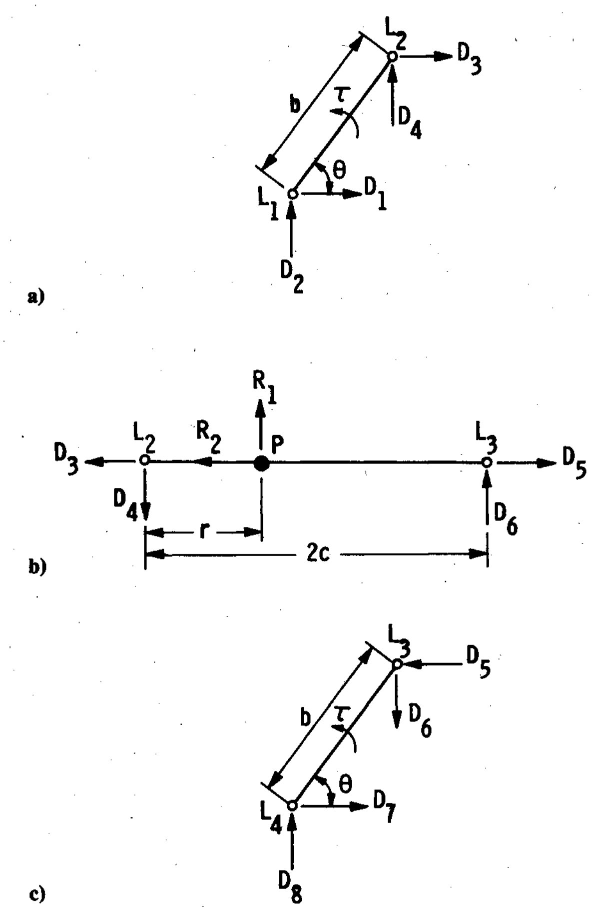

Fig. 2 Free-body diagrams used with momentum principles.

respectively; and the use of the linear momentum principle in conjunction with Fig. 2b produces

$$R_1 - D_4 + D_6 = ma_1^P \tag{16}$$

$$R_2 + D_3 - D_5 = ma_2^P \tag{17}$$

where a_1^P and a_2^P are the $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ measure numbers of the acceleration of P in N. Elimination of $D_3,\ldots,D_6$ from Eqs. (13)-(17) leads to

$$mb(a_1^P\cos\theta + a_2^P\sin\theta) = 2\tau + b(R_1\cos\theta + R_2\sin\theta) \tag{18}$$

and application of the linear momentum principle to P alone gives

$$\sigma + R_2 = ma_2^P \tag{19}$$

All that remains to be done is to express a_1^P and a_2^P in terms of ω_1, ω_2, ω_3, v_1, v_2, v_3, r, θ, and time-derivatives of these quantities. After doing this, one arrives at the last two dynamical differential equations,

$$\begin{aligned} mb\{ & [\dot v_1 + \omega_2 v_3 - \omega_3 v_2 - b\ddot\theta\cos\theta - (\dot\omega_3 - \omega_1\omega_2)(c - r - b\cos\theta) \\ & - 2\omega_3(b\dot\theta\sin\theta - \dot r) - (\omega_2^2 + \omega_3^2)b\sin\theta - b\dot\theta^2\sin\theta]\cos\theta \\ & + [\dot v_2 + \omega_3 v_1 - \omega_1 v_3 + b\ddot\theta\sin\theta + b\dot\theta^2\cos\theta - \ddot r \\ & + (\dot\omega_3 + \omega_1\omega_2)b\sin\theta + 2\omega_3 b\dot\theta\cos\theta \\ & - (\omega_3^2 + \omega_1^2)(c - r - b\cos\theta)]\sin\theta\} \\ & = 2\tau + b(R_1\cos\theta + R_2\sin\theta) \end{aligned} \tag{20}$$

and

$$\begin{aligned} m[& \dot v_2 + \omega_3 v_1 - \omega_1 v_3 + b\ddot\theta\sin\theta + b\dot\theta^2\cos\theta - \ddot r + (\dot\omega_3 + \omega_1\omega_2)b\sin\theta \\ & + 2\omega_3 b\dot\theta\cos\theta - (\omega_3^2 + \omega_1^2)(c - r - b\cos\theta)] = \sigma + R_2 \end{aligned} \tag{21}$$

In principle, the fourteen equations (1), (2), (6-8), (10-12), (20), and (21) suffice for the solution of the initial value problem under consideration. Eqs. (1) and (2), involving explicit expressions for first time-derivatives of dependent variables, are in a form well suited for numerical integration, but Eqs. (6-8), (10-12), (20), and (21) require further work. Specifically, $\ddot r$ and $\ddot\theta$ must be eliminated, a task easily accomplished by introducing two auxiliary dependent variables, say x and y, as $x \triangleq \dot r$ and $y \triangleq \dot\theta$; the dynamical equations must then be solved for $\dot\omega_1$, $\dot\omega_2$, $\dot\omega_3$, $\dot v_1$, $\dot v_2$, $\dot v_3$, $\dot x$, and $\dot y$, which can be done either analytically or at each step of a numerical integration.

As regards the labor performed to generate the equations, we note two drawbacks of the method at hand. One is that it necessitates the introduction and subsequent elimination of certain constraint forces, namely $D_3,\ldots,D_6$; the other is that, whenever one uses the angular momentum principle, one is forced to locate the mass center of the system to which the principle is being applied [S^* in the case of S, and P in Fig. 2b], a point that need not be located otherwise. The concomitant effort would become especially noticeable if the linkage were not regarded as massless, for Eqs. (13-15) would then involve also D_1, D_2, D_7, and D_8, which would necessitate writing four additional equations, as well as subsequently eliminating the new unknowns.

IV. D'Alembert's Principle

D'Alembert's principle justifies the following assertion: If the particles of a set are acted upon by body and contact forces and each particle is regarded as being subjected also to the action of an "inertia force," this force being defined as a vector whose line of action passes through the particle and whose magnitude and direction are found by multiplying the mass of the particle with the negative of the inertial acceleration of the particle, then the system of all body forces, contact forces, and inertia forces is a null system; that is, the sum of the forces and the sum of the moments of the forces about *any* point vanish. To apply this idea effectively to the formulation of equations of motion for systems containing rigid bodies one needs to know that the system of all inertia forces acting on a rigid body A can be replaced with a couple together with a force applied at the mass center of A, and that the associated torque $\boldsymbol{T}^*$ and force $\boldsymbol{F}^*$, called, respectively, the inertia torque and the inertia force for the body, can be expressed as

$$\boldsymbol{T}^* = -(\boldsymbol{I}\cdot\boldsymbol{\alpha} + \boldsymbol{\omega}\times\boldsymbol{I}\cdot\boldsymbol{\omega}) \qquad \boldsymbol{F}^* = -M\boldsymbol{a} \tag{22}$$

where $\boldsymbol{I}$ is the central inertia dyadic of A, $\boldsymbol{\alpha}$ and $\boldsymbol{\omega}$ are the inertial angular acceleration and angular velocity of A, M is the mass of A, and $\boldsymbol{a}$ is the inertial acceleration of the mass center of A.

Equating to zero the sum of all forces (including inertia forces) acting on S, one arrives once again at Eqs. (6-8); but summing moments of all forces about A^* and setting the result equal to zero yields equations differing from Eqs. (10-12), namely

$$\begin{aligned} & (I_2 - I_3)\omega_2\omega_3 - I_1\dot\omega_1 + T_1 + R_3(c - r - b\cos\theta) \\ & - m(c - r - b\cos\theta)[\dot v_3 + \omega_1 v_2 - \omega_2 v_1 \\ & + (\dot\omega_1 + \omega_2\omega_3)(c - r - b\cos\theta) \\ & - (\dot\omega_2 - \omega_3\omega_1)b\sin\theta + 2b\dot\theta(\omega_1\sin\theta - \omega_2\cos\theta) - 2\dot r\omega_1] = 0 \end{aligned} \tag{23}$$

$$(I_3-I_1)\omega_3\omega_1-I_2\dot{\omega}_2+T_2-R_3b\sin\theta$$
$$+mb\sin\theta[\dot{v}_3+\omega_1v_2-\omega_2v_1+(\dot{\omega}_1+\omega_2\omega_3)(c-r-b\cos\theta)$$
$$-(\dot{\omega}_2-\omega_3\omega_1)b\sin\theta+2b\dot{\theta}(\omega_1\sin\theta-\omega_2\cos\theta)-2\dot{r}\omega_1]=0 \tag{24}$$

$$(I_1-I_2)\omega_1\omega_2-I_3\dot{\omega}_3+T_3+R_2b\sin\theta-R_1(c-r-b\cos\theta)$$
$$-m\{b\sin\theta[\dot{v}_2+\omega_3v_1-\omega_1v_3+(\ddot{\theta}+\dot{\omega}_3+\omega_1\omega_2)b\sin\theta$$
$$+(\dot{\theta}+2\omega_3)b\dot{\theta}\cos\theta-\ddot{r}-\omega_1^2(c-r-b\cos\theta)]$$
$$-(c-r-b\cos\theta)[\dot{v}_1+\omega_2v_3-\omega_3v_2+b\ddot{\theta}\cos\theta$$
$$-(\dot{\omega}_3-\omega_1\omega_2)(c-r-b\cos\theta)-2\omega_3(b\dot{\theta}\sin\theta-\dot{r})$$
$$-(\dot{\theta}^2+\omega_1^2)b\sin\theta]\}=0 \tag{25}$$

There are two obvious differences between these last three equations and Eqs. (10-12). Equations (10-12) contain S_1, S_2, S_3, and μ, whereas Eqs. (23-25) do not; and Eqs. (23-25) possess terms involving v_1, v_2, v_3 and their first derivatives, whereas no such terms appear in Eqs. (10-12). Speaking broadly, one can say that both differences are attributable directly to the fact that D'Alembert's principle leaves one free to take moments about *any* point, whereas the angular momentum principle restricts one in this regard. More specifically, by taking moments about A^* rather than S^* we have eliminated the appearance of $mM/(m+M)$, that is, μ, as well as any contributions from the force $S_1\mathbf{a}_1+S_2\mathbf{a}_2+S_3\mathbf{a}_3$. However, the moment about A^* of the inertia force acting on P involves the quantities v_1, v_2, v_3, $\dot{v}_1$, $\dot{v}_2$, $\dot{v}_3$, which do not appear when one employs the angular momentum principle. The effort involved in bringing these terms into the rotational equations is more than offset by the considerable savings in labor resulting from the use of D'Alembert's principle rather than the angular momentum principle.

As before, two further dynamical equations remain to be derived. Once again, free-body diagrams are found to be helpful, but these now include inertia forces, as shown in Figs. 3a-3c, the first of which is identical to Fig. 2a because no inertia forces come into play in connection with the (massless) link L_1-L_2. In Figs. 3b and 3c, the quantities a_1^P and a_2^P have the same meaning as in Eqs. (16) and (17).

Referring to Fig. 3a, one can write

$$D_2-D_4=0 \tag{26}$$

and taking moments about point L_3 in Fig. 3b gives

$$2cD_4-(2c-r)(R_1-ma_1^P)=0 \tag{27}$$

while taking moments about L_4 in Fig. 3c produces

$$2cD_2-b\cos\theta(R_2-ma_2^P)$$
$$+(2c-r-b\sin\theta)(R_1-ma_1^P)-2\tau=0 \tag{28}$$

so that, after eliminating D_2 and D_4, one recovers Eq. (18).

Although this method, like the use of momentum principles, requires the introduction and subsequent elimination of constraint forces, it permits one to arrive at results with less labor because the number of equations one needs to write is smaller if one makes judicious choices of moment centers. This advantage would become all the more significant if the linkage were regarded as massive because doing so would not complicate the procedure excessively. The number of terms appearing in each of Eqs. (26-28) would increase, but the number of unknowns and equations would remain unaltered.

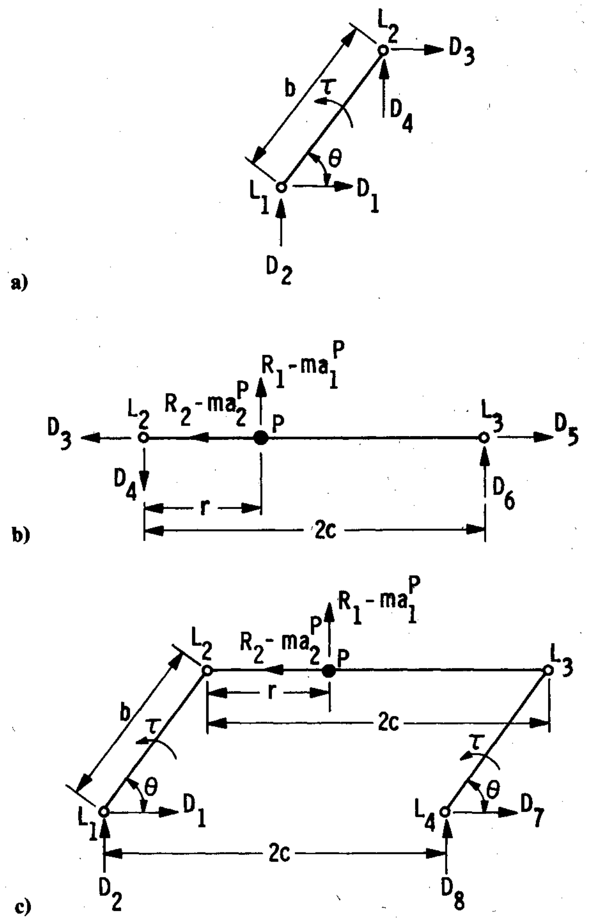

Fig. 3 Free-body diagrams used with D'Alembert's principle.

Similarly, one could add the necessary terms to Eqs. (23-25) without being compelled to determine the location of S^*, a task that could not be avoided if the angular momentum principle were used.

V. Lagrange's Equations

To formulate Lagrange's equations, one needs T, the kinetic energy of the spacecraft S, which can be written

$$T=(M/2)(v_1^2+v_2^2+v_3^2)+(m/2)\{[v_1+b\dot{\theta}\cos\theta$$
$$-\omega_3(c-r-b\cos\theta)]^2+(v_2+b\dot{\theta}\sin\theta$$
$$-\dot{r}+\omega_3b\sin\theta)^2+[v_3+\omega_1(c-r-b\cos\theta)$$
$$-\omega_2b\sin\theta]^2\}+(I_1\omega_1^2+I_2\omega_2^2+I_3\omega_3^2)/2 \tag{29}$$

Because ω_1, ω_2, ω_3, and v_1, v_2, v_3 all are independent of r, θ, $\dot{r}$, and $\dot{\theta}$, two dynamical equations can be formed by operating on T as written and equating the results to appropriate generalized forces; i.e.,

$$\frac{d}{dt}\frac{\partial T}{\partial\dot{\theta}}-\frac{\partial T}{\partial\theta}=Q_\theta \tag{30}$$

and

$$\frac{d}{dt}\frac{\partial T}{\partial\dot{r}}-\frac{\partial T}{\partial r}=Q_r \tag{31}$$

The generalized forces Q_θ and Q_r are given by

$$Q_\theta=-\sigma-R_2 \tag{32}$$

and

$$Q_r = 2\tau + b(R_1 \cos\theta + R_2 \sin\theta) \tag{33}$$

and Eqs. (30) and (31) lead directly to Eqs. (20) and (21), respectively, doing so without requiring the use of free-body diagrams or the introduction and subsequent elimination of the constraint forces $D_1,...,D_8$. Moreover, the effort to develop these equations would not be increased significantly if the links were treated as massive. However, despite these advantages of the present method over the preceding one, Lagrange's equations are soon seen to be quite unsatisfactory when one attempts to use them to derive equations to replace Eqs. (6-8) and (10-12), that is, when one operates on T in accordance with

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = Q_i \qquad (i = 1,...,6) \tag{34}$$

Consider, for example, the equation corresponding to $i = 1$. Referring to Eq. (29), one has

$$\frac{\partial T}{\partial \dot{q}_1} = m\{ -[v_1 + b\dot{\theta}\cos\theta - \omega_3(c - r - b\cos\theta)]s_2(c - r - b\cos\theta) + (v_2 + b\dot{\theta}\sin\theta - \dot{r} + \omega_3 b\sin\theta)s_2 b\sin\theta$$

$$+ [v_3 + \omega_1(c - r - b\cos\theta) - \omega_2 b\sin\theta][c_2c_3(c - r - b\cos\theta) + c_2s_3 b\sin\theta]\} + I_1\omega_1c_2c_3 - I_2\omega_2c_2s_3 + I_3\omega_3s_2 \tag{35}$$

and

$$\frac{\partial T}{\partial q_1} = M[v_1(\dot{q}_6C_{21} - \dot{q}_5C_{31}) + v_2(\dot{q}_6C_{22} - \dot{q}_5C_{32}) + v_3(\dot{q}_6C_{23} - \dot{q}_5C_{33})] + m\{[v_1 + b\dot{\theta}\cos\theta - \omega_3(c - r - b\cos\theta)](\dot{q}_6C_{21} - \dot{q}_5C_{31})$$

$$+ (v_2 + b\dot{\theta}\sin\theta - \dot{r} + \omega_3 b\sin\theta)(\dot{q}_6C_{22} - \dot{q}_5C_{32}) + [v_3 + \omega_1(c - r - b\cos\theta) - \omega_2 b\sin\theta](\dot{q}_6C_{23} - \dot{q}_5C_{33})\} \tag{36}$$

where

$$C_{ij} \triangleq n_i \cdot a_j \qquad (i,j = 1,2,3) \tag{37}$$

so that, for instance, $C_{32} = c_1s_2s_3 + c_3s_1$. Differentiating Eq. (35) with respect to t gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}_1} = m\{ -[\dot{v}_1 + b(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) - \dot{\omega}_3(c - r - b\cos\theta) - \omega_3(b\dot{\theta}\sin\theta - \dot{r})]s_2(c - r - b\cos\theta)$$

$$+ [v_1 + b\dot{\theta}\cos\theta - \omega_3(c - r - b\cos\theta)][\dot{q}_2c_2(c - r - b\cos\theta) + s_2(b\dot{\theta}\sin\theta - \dot{r})]$$

$$+ [\dot{v}_2 + b(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta) - \ddot{r} + \dot{\omega}_3 b\sin\theta + \omega_3 b\dot{\theta}\cos\theta]s_2 b\sin\theta + (v_2 + b\dot{\theta}\sin\theta - \dot{r} + \omega_3 b\sin\theta)b(\dot{q}_2c_2\sin\theta + \dot{\theta}s_2\cos\theta)$$

$$+ [\dot{v}_3 + \dot{\omega}_1(c - r - b\cos\theta) + \omega_1(b\dot{\theta}\sin\theta - \dot{r}) - \dot{\omega}_2 b\sin\theta - \omega_2\dot{\theta}b\cos\theta][c_2c_3(c - r - b\cos\theta) + c_2s_3 b\sin\theta]$$

$$+ [v_3 + \omega_1(c - r - b\cos\theta) - \omega_2 b\sin\theta][-(\dot{q}_2s_2c_3 + \dot{q}_3c_2s_3)(c - r - b\cos\theta) + c_2c_3(b\dot{\theta}\sin\theta - \dot{r})$$

$$+ (-\dot{q}_2s_2s_3 + \dot{q}_3c_2c_3)b\sin\theta + c_2s_3 b\dot{\theta}\cos\theta]\}$$

$$+ I_1[\dot{\omega}_1c_2c_3 - \omega_1(\dot{q}_2s_2c_3 + \dot{q}_3c_2s_3)] - I_2[\dot{\omega}_2c_2s_3 + \omega_2(-\dot{q}_2s_2s_3 + \dot{q}_3c_2c_3)] + I_3(\dot{\omega}_3s_2 + \omega_3\dot{q}_2c_2) \tag{38}$$

To complete the task of writing the Lagrange equation for $i = 1$, one must form the generalized force Q_1, which turns out to be given by

$$Q_1 = s_2[T_3 + bR_2\sin\theta - (c - r - b\cos\theta)R_1] + \{c_2c_3[T_1 + (c - r - b\cos\theta)R_3] + c_2s_3(bR_3\sin\theta - T_2)\} \tag{39}$$

and then one must substitute from Eqs. (36), (38), and (39) into Eq. (34). Next, to bring the resulting equation into a form suitable for numerical solution, one is forced to deal with the presence of second time-derivatives of r and θ, and first time-derivatives of q_2, q_3, q_5, q_6. The former can be eliminated by making use of the auxiliary dependent variables x and y introduced as in connection with the momentum principles, and the corresponding difficulty associated with the $\dot{q}$'s may be surmounted by using Eqs. (1) and (2) whenever necessary. Thus, one is able to make the following replacements:

$$\dot{q}_6C_{21} - \dot{q}_5C_{31} = v_2C_{13} - v_3C_{12} \tag{40a}$$

$$\dot{q}_6C_{22} - \dot{q}_5C_{32} = v_3C_{11} - v_1C_{13} \tag{40b}$$

$$\dot{q}_6C_{23} - \dot{q}_5C_{33} = v_1C_{12} - v_2C_{11} \tag{40c}$$

$$\dot{q}_2s_2c_3 + \dot{q}_3c_2s_3 = \omega_2C_{13} - \omega_3C_{12} \tag{40d}$$

$$-\dot{q}_2s_2s_3 + \dot{q}_3c_2c_3 = \omega_3C_{11} - \omega_1C_{13} \tag{40e}$$

$$\dot{q}_2c_2 = \omega_2C_{11} - \omega_1C_{12} \tag{40f}$$

Finally, to obtain the complete set of dynamical equations intended to replace Eqs. (6-8) and (10-12), one must repeat five more times all the steps analogous to those involved in the writing of Eqs. (35-40). This is a formidable undertaking. Furthermore, comparison of Eq. (6) with the Lagrange equation for $i=1$ reveals that equations of motion produced by the Lagrange method are much lengthier than those obtained by applying momentum principles or D'Alembert's principle. This is not a spurious result brought about by failing to cancel terms in Eqs. (36) and (38), as can be seen by noting, for example, that $\dot\omega_1$, $\dot\omega_2$, and $\dot\omega_3$ all appear in Eq. (38), whereas only $\dot\omega_3$ occurs in Eq. (6).

The shortcomings of the Lagrange method become particularly evident when it is noted that we have here used Lagrange's equations in an especially efficient way by expressing T as in Eq. (29) and then retaining ω_1, ω_2, ω_3, v_1, v_2, v_3 in Eqs. (36) and (38) wherever possible. Were one to use Lagrange's equations more naively, that is, by first solving Eqs. (1) and (2) for ω_1, ω_2, ω_3, v_1, v_2, v_3 to obtain

$$\begin{aligned}
\omega_1 &= \dot q_1 c_2 c_3 + \dot q_2 s_3 \\
\omega_2 &= -\dot q_1 c_2 s_3 + \dot q_2 c_3 \\
\omega_3 &= \dot q_1 s_2 + \dot q_3 \\
v_1 &= \dot q_4 c_2 c_3 + \dot q_5 (s_1 s_2 c_3 + s_3 c_1) + \dot q_6 (-c_1 s_2 c_3 + s_3 s_1) \\
v_2 &= -\dot q_4 c_2 s_3 + \dot q_5 (-s_1 s_2 s_3 + c_3 c_1) + \dot q_6 (c_1 s_2 s_3 + c_3 s_1) \\
v_3 &= \dot q_4 s_2 - \dot q_5 s_1 c_2 + \dot q_6 c_1 c_2
\end{aligned} \tag{41}$$

and then substituting these expressions into Eq. (29) before performing the differentiations required by Eq. (34), one would arrive at equations even longer than those reported here. The reader may verify this by substituting from Eqs. (41) into Eqs. (36) and (38).

VI. Hamilton's Canonical Equations

The writing of Hamilton's canonical equations for a system possessing n degrees of freedom involves $2n$ partial differentiations of the Hamiltonian H, where H is regarded as a function of the generalized coordinates $q_1,\ldots,q_n$ and generalized momenta $p_1,\ldots,p_n$. Since H is given by

$$H = \sum_{i=1}^{n} p_i \dot q_i - L \tag{42}$$

where L is the Lagrangian of the system, one therefore begins by seeking expressions for $\dot q_1,\ldots,\dot q_n$ in terms of $p_1,\ldots,p_n$ in order to eliminate $\dot q_1,\ldots,\dot q_n$ from Eq. (42). To this end, one forms the generalized momenta $p_i = \partial T/\partial \dot q_i$ $(i=1,\ldots,n)$, where T is the kinetic energy of the system. If q_7 and q_8 are taken to be r and θ, respectively, this leads in our example to

$$\begin{aligned}
p_1 = m\{ & -[\dot q_4 C_{11} + \dot q_5 C_{21} + \dot q_6 C_{31} + b\dot q_8 c_8 \\
& -(\dot q_1 s_2 + \dot q_3)(c - q_7 - bc_8)] s_2 (c - q_7 - bc_8) \\
& + [\dot q_4 C_{12} + \dot q_5 C_{22} + \dot q_6 C_{32} + b\dot q_8 s_8 - \dot q_7 \\
& + (\dot q_1 s_2 + \dot q_3) b s_8] s_2 b s_8 + [\dot q_4 C_{13} + \dot q_5 C_{23} \\
& + \dot q_6 C_{33} + (\dot q_1 c_2 c_3 + \dot q_2 s_3)(c - q_7 - bc_8) \\
& - (-\dot q_1 c_2 s_3 + \dot q_2 c_3) b s_8] [c_2 c_3 (c - q_7 - bc_8) \\
& + c_2 s_3 b s_8] \} + I_1 (\dot q_1 c_2 c_3 + \dot q_2 s_3) c_2 c_3 \\
& - I_2 (-\dot q_1 c_2 s_3 + \dot q_2 c_3) c_2 s_3 + I_3 (\dot q_1 s_2 + \dot q_3) s_2
\end{aligned} \tag{43}$$

	1	2	3	4	5	6	7	8
1	X	X	X	X	X	X	X	X
2	X	X		X	X	X		
3	X		X	X	X	X	X	X
4	X	X	X	X			X	X
5	X	X	X		X		X	X
6	X	X	X			X	X	X
7	X		X	X	X	X	X	X
8	X		X	X	X	X	X	X

Fig. 4 Nonzero element locations.

where s_8 and c_8 stand for $\sin q_8$ and $\cos q_8$, respectively; and similar expressions are found for $p_2,\ldots,p_8$. The resulting set of eight equations, which is linear in $\dot q_1,\ldots,\dot q_8$, must be solved for these quantities, a task equivalent to the inversion *in literal form* of an 8×8 matrix which, it turns out, has nonzero elements in each of the places indicated by a cross in Fig. 4. Clearly, this is an undertaking so laborious as to be essentially prohibitive. Hence, this approach does not even permit us to write equations of motion, so that questions regarding the form of the equations of motion cannot be addressed.

To overcome the difficulty encountered here, one can attempt to work with implicit functions and thus relegate certain inescapable matrix inversions to a later stage of the analysis, where they can be performed numerically, rather than literally. This is possible in principle, but leads ultimately to an algorithm so complex as to be unacceptable.

VII. Boltzmann-Hamel Equations

The Boltzmann-Hamel equations* permit one to work with dependent variables other than generalized coordinates and generalized momenta, such as angular velocity and translational velocity measure numbers, and at the same time to enjoy the chief benefit of the Lagrangian approach, namely automatic elimination of certain interaction forces. To form the equations, one begins by defining the dependent variables in terms of $q_1,\ldots,q_n$. In our example, this amounts to introducing $u_1,\ldots,u_8$ as

$$\begin{aligned}
& u_1 \triangleq \dot q_1 c_2 c_3 + \dot q_2 s_3 \qquad u_2 \triangleq -\dot q_1 c_2 s_3 + \dot q_2 c_3 \qquad u_3 \triangleq \dot q_1 s_2 + \dot q_3 \\
& u_4 \triangleq \dot q_4 c_2 c_3 + \dot q_5 (s_1 s_2 c_3 + s_3 c_1) + \dot q_6 (-c_1 s_2 c_3 + s_3 s_1) \\
& u_5 \triangleq -\dot q_4 c_2 s_3 + \dot q_5 (-s_1 s_2 s_3 + c_3 c_1) + \dot q_6 (c_1 s_2 s_3 + c_3 s_1) \\
& u_6 \triangleq \dot q_4 s_2 - \dot q_5 s_1 c_2 + \dot q_6 c_1 c_2 \qquad u_7 \triangleq b\dot q_8 \qquad u_8 \triangleq b\dot q_8 s_8 - \dot q_7
\end{aligned} \tag{44}$$

the first six being the a_1, a_2, a_3 measure numbers of ω and v^{A^*}, while the last two are chosen so as to bring the velocity of P in A into a particularly simple form,† namely $u_7 c_8 a_1 + u_8 a_2$. Equations (44) are then solved for $\dot q_1,\ldots,\dot q_8$:

$$\begin{aligned}
& \dot q_1 = (u_1 c_3 - u_2 s_3)/c_2 \qquad \dot q_2 = u_1 s_3 + u_2 c_3 \\
& \dot q_3 = (-u_1 c_3 + u_2 s_3) s_2 / c_2 + u_3 \qquad \dot q_4 = u_4 c_2 c_3 - u_5 c_2 s_3 + u_6 s_2 \\
& \dot q_5 = u_4 (s_1 s_2 c_3 + s_3 c_1) + u_5 (-s_1 s_2 s_3 + c_3 c_1) - u_6 s_1 c_2 \\
& \dot q_6 = u_4 (-c_1 s_2 c_3 + s_3 s_1) + u_5 (c_1 s_2 s_3 + c_3 s_1) + u_6 c_1 c_2 \\
& \dot q_7 = u_7 s_8 - u_8 \qquad \dot q_8 = u_7 / b
\end{aligned} \tag{45}$$

*These equations are called by some authors Lagrange's quasi-coordinate equations.

†One could define u_7 as $u_7 \triangleq = b\dot q_8 c_8$, which would lead to an even simpler expression for the velocity of P, but this defining relation could not be solved for $\dot q_8$ when $q_8 = \pi/2$.

and 128 quantities α_{ij} and β_{ij} $(i,j=1,...,8)$ are identified by comparing Eqs. (44) and (45) with

$$u_j=\sum_{i=1}^{8}\alpha_{ij}\dot{q}_i \qquad (j=1,...,8) \tag{46}$$

and

$$\dot{q}_i=\sum_{j=1}^{8}\beta_{ij}u_j \qquad (i=1,...,8) \tag{47}$$

respectively, which produces the following nonvanishing functions of $q_1,...,q_8$:

$$\alpha_{11}=c_2c_3,\ \alpha_{21}=s_3,\ \alpha_{12}=-c_2s_3,\ \alpha_{22}=c_3,\ \alpha_{13}=s_2,\ \alpha_{33}=1,$$
$$\alpha_{44}=c_2c_3,\ \alpha_{54}=s_1s_2c_3+s_3c_1,\ \alpha_{64}=-c_1s_2c_3+s_3s_1,$$
$$\alpha_{45}=-c_2s_3,\ \alpha_{55}=-s_1s_2s_3+c_3c_1,\ \alpha_{65}=c_1s_2s_3+c_3s_1,$$
$$\alpha_{46}=s_2,\ \alpha_{56}=-s_1c_2,\ \alpha_{66}=c_1c_2$$
$$\alpha_{87}=b,\ \alpha_{78}=-1,\ \alpha_{88}=bs_8 \tag{48}$$

$$\beta_{11}=c_3/c_2,\ \beta_{12}=-s_3/c_2,\ \beta_{21}=s_3,\ \beta_{22}=c_3,\ \beta_{31}=-c_3s_2/c_2,$$
$$\beta_{32}=s_2s_3/c_2,\ \beta_{33}=1,\ \beta_{44}=c_2c_3,\ \beta_{45}=-c_2s_3,$$
$$\beta_{46}=s_2,\ \beta_{54}=s_1s_2c_3+s_3c_1,\ \beta_{55}=-s_1s_2s_3+c_3c_1,$$
$$\beta_{56}=-s_1c_2,\ \beta_{64}=-c_1s_2c_3+s_3s_1,\ \beta_{65}=c_1s_2s_3+c_3s_1,$$
$$\beta_{66}=c_1c_2,\ \beta_{77}=s_8,\ \beta_{78}=-1,\ \beta_{87}=1/b \tag{49}$$

Next, 512 quantities γ_{ijk} are formed in accordance with

$$\gamma_{ijk}\triangleq\sum_{r=1}^{8}\sum_{s=1}^{8}\beta_{ri}\beta_{sk}\left(\frac{\partial\alpha_{rj}}{\partial q_s}-\frac{\partial\alpha_{sj}}{\partial q_r}\right) \qquad (i,j,k=1,...,8) \tag{50}$$

For instance

$$\gamma_{156}=\sum_{r=1}^{8}\sum_{s=1}^{8}\beta_{r1}\beta_{s6}\left(\frac{\partial\alpha_{r5}}{\partial q_s}-\frac{\partial\alpha_{s5}}{\partial q_r}\right) \tag{51}$$

or, since only $r=1,2,3$ and $s=4,5,6$ give rise to nonzero β's [see Eqs. (49)] and α_{15}, α_{25}, α_{35} are all equal to zero,

$$\begin{aligned}\gamma_{156}=&-\beta_{11}[\beta_{46}(\partial\alpha_{45}/\partial q_1)+\beta_{56}(\partial\alpha_{55}/\partial q_1)+\beta_{66}(\partial\alpha_{65}/\partial q_1)]\\&-\beta_{21}[\beta_{46}(\partial\alpha_{45}/\partial q_2)+\beta_{56}(\partial\alpha_{55}/\partial q_2)+\beta_{66}(\partial\alpha_{65}/\partial q_2)]\\&-\beta_{31}[\beta_{46}(\partial\alpha_{45}/\partial q_3)+\beta_{56}(\partial\alpha_{55}/\partial q_3)+\beta_{66}(\partial\alpha_{65}/\partial q_3)]\end{aligned} \tag{52}$$

which, once the indicated differentiations have been carried out by reference to Eqs. (48), finally reduces to $\gamma_{156}=-1$.

After performing similar steps 511 more times, one writes T in terms of $q_1,...,q_8$ and $u_1,...,u_8$, obtaining

$$\begin{aligned}T=&(M/2)(u_4^2+u_5^2+u_6^2)+(m/2)\{[u_4+u_7c_8\\&-u_3(c-q_7-bc_8)]^2+(u_5+u_8+u_3bs_8)^2\\&+[u_6+u_1(c-q_7-bc_8)-u_2bs_8]^2\}+(I_1u_1^2+I_2u_2^2+I_3u_3^2)/2\end{aligned} \tag{53}$$

and then one must form $\delta_1,...,\delta_8$ and $\epsilon_1,...,\epsilon_8$ as

$$\delta_i\triangleq\sum_{j=1}^{8}\beta_{ji}\left(\frac{\partial T}{\partial q_j}\right) \qquad \epsilon_i\triangleq\frac{\partial T}{\partial u_i} \qquad (i=1,...,8) \tag{54}$$

obtaining

$$\delta_1=\delta_2=...=\delta_6=0 \tag{55a}$$

$$\begin{aligned}\delta_7=m\langle &s_8\{u_3[u_4+u_7c_8-u_3(c-q_7-bc_8)]\\&-u_1[u_6+u_1(c-q_7-bc_8)-u_2bs_8]\}\\&-s_8(u_7/b+u_3)[u_4+u_7c_8-u_3(c-q_7-bc_8)]\\&+u_3c_8(u_5+u_8+u_3bs_8)+(u_1s_8-u_2c_8)[u_6\\&+u_1(c-q_7-bc_8)-u_2bs_8]\rangle\end{aligned} \tag{55b}$$

$$\begin{aligned}\delta_8=&-m\{u_3[u_4+u_7c_8-u_3(c-q_7-bc_8)]\\&-u_1[u_6+u_1(c-q_7-bc_8)-u_2bs_8]\}\end{aligned} \tag{55c}$$

and

$$\begin{aligned}&\epsilon_1=m[u_6+u_1(c-q_7-bc_8)-u_2bs_8](c-q_7-bc_8)+I_1u_1\\&\epsilon_2=-m[u_6+u_1(c-q_7-bc_8)-u_2bs_8]+I_2u_2\\&\epsilon_3=m\{-[u_4+u_7c_8-u_3(c-q_7-bc_8)](c-q_7-bc_8)\\&\qquad+(u_5+u_8+u_3bs_8)bs_8\}+I_3u_3\\&\epsilon_4=Mu_4+m[u_4+u_7c_8-u_3(c-q_7-bc_8)]\\&\epsilon_5=Mu_5+m(u_5+u_8+u_3bs_8)\\&\epsilon_6=Mu_6+m[u_6+u_1(c-q_7-bc_8)-u_2bs_8]\\&\epsilon_7=mc_8[u_4c_8-u_3(c-q_7-bc_8)]\\&\epsilon_8=m(u_5+u_8+u_3bs_8)\end{aligned} \tag{56}$$

which brings one into position to write the left-hand members of the Boltzmann-Hamel equations, these equations being

$$\dot{\epsilon}_i+\sum_{j=1}^{8}\sum_{k=1}^{8}\gamma_{ijk}u_k\epsilon_j-\delta_i=\sum_{j=1}^{8}\beta_{ji}Q_j \qquad (i=1,...,8) \tag{57}$$

where $Q_1,...,Q_8$ are the generalized forces associated with $q_1,...,q_8$, respectively. Once these have been formed, one finally obtains, for $i=1$,

$$\begin{aligned}&(I_2-I_3)u_2u_3-I_1\dot{u}_1+T_1+R_3(c-q_7-bc_8)\\&-m(c-q_7-bc_8)[\dot{u}_6+\dot{u}_1(c-q_7-bc_8)\\&-\dot{u}_2bs_8+2(u_1u_8-u_2u_7c_8)+u_1u_5-u_2u_4\\&+u_3u_1bs_8+u_2u_3(c-q_7-bc_8)]=0\end{aligned} \tag{58}$$

which is very similar to Eq. (23), the difference between the two being that u_1, u_2, u_3 appear in place of ω_1, ω_2, ω_3, respectively; and u_7 and u_8 represent $b\dot{\theta}$ and $b\dot{\theta}\sin\theta-\dot{r}$, respectively, which is why Eq. (58) is a little shorter than Eq. (23). Similarly, for $i=2,...,8$, equations equivalent to ones already found by other methods are obtained with much effort.

The fact that the Boltzmann-Hamel equations are equivalent to those resulting from the use of other methods is of interest only insofar as it shows that the equations have been written correctly. The real issue is whether or not they are simpler in form and, if so, whether or not the obviously great amount of labor expended in their formulation is justified by such improvements in form as they may represent.

That this labor truly is great will not escape the notice of anyone who undertakes the task of writing these equations.

In the light of the observations made about the Boltzmann-Hamel equations, the fact that these equations are ever used at all may seem surprising. However, there exist situations in which one can employ them without becoming totally mired in a profusion of γ's, and in such situations the benefits the Boltzmann-Hamel equations offer are at least purchased for a reasonable price. For example, this is the case when the system under consideration is a rigid body with one fixed point.[9] The difficulty is that one generally cannot know a priori whether or not such a simplification is possible.

VIII. Gibbs Equations

Use of the Gibbs equations‡ offers one the same advantages as does use of the Boltzmann-Hamel equations, but frees one from the necessity to form the troublesome quantities γ_{ijk} [see Eq. (50)], for one now works with

$$\frac{\partial G}{\partial \dot{u}_i} = \sum_{j=1}^{n} \beta_{ji} Q_j \qquad (i=1,\ldots,n) \tag{59}$$

where β_{ji} and Q_j have the same meaning as before and the Gibbs function G for a set S of particles $P_1,\ldots,P_\nu$ is defined in terms of the mass m_i and the inertial acceleration a_i of P_i $(i=1,\ldots,\nu)$ as

$$G=\frac{1}{2}\sum_{i=1}^{\nu} m_i a_i^2 \tag{60}$$

Moreover, when constructing G_A, the relevant contribution to G of a subset of S that forms a rigid body A, one can employ the relationship

$$G_A = \tfrac{1}{2}(Ma^2 + \boldsymbol{\alpha}\cdot \boldsymbol{I}\cdot\boldsymbol{\alpha} + 2\boldsymbol{\alpha}\cdot\boldsymbol{\omega}\times\boldsymbol{I}\cdot\boldsymbol{\omega}) \tag{61}$$

where M is the mass of A, and $\boldsymbol{a}$, $\boldsymbol{\alpha}$, $\boldsymbol{\omega}$, and $\boldsymbol{I}$ are the inertial acceleration of the mass center of A, the inertial angular acceleration of A, the inertial angular velocity of A, and the central inertia dyadic of A, respectively.

With the aid of Eq. (61), one finds that G for the spacecraft S is given by

$$\begin{aligned} G= {} & (M/2)\,[\,(\dot{u}_4+u_2u_6-u_3u_5)^2+(\dot{u}_5+u_3u_4-u_1u_6)^2+(\dot{u}_6+u_1u_5-u_2u_4)^2\,]+(m/2)\Big\langle\{\dot{u}_4+\dot{u}_7c_8-u_7^2s_8/b \\ & -\dot{u}_3(c-q_7-bc_8)-u_3u_8+u_2[u_6+u_1(c-q_7-bc_8)-u_2bs_8]-u_3(u_5+u_8+u_3bs_8)\}^2+\{\dot{u}_5+\dot{u}_8+\dot{u}_3bs_8+u_3u_7c_8 \\ & +u_3[u_4+u_7c_8-u_3(c-q_7-bc_8)]-u_1[u_6+u_1(c-q_7-bc_8)-u_2bs_8]\}^2+\{\dot{u}_6+\dot{u}_1(c-q_7-bc_8)+u_1u_8-\dot{u}_2bs_8-u_2u_7c_8 \\ & +u_1(u_5+u_8+u_3bs_8)-u_2[u_4+u_7c_8-u_3(c-q_7-bc_8)]\}^2\Big\rangle+(I_1\dot{u}_1^2+I_2\dot{u}_2^2+I_3\dot{u}_3^2)/2-\dot{u}_1u_2u_3(I_2-I_3) \\ & -\dot{u}_2u_3u_1(I_3-I_1)-\dot{u}_3u_1u_2(I_1-I_2) \end{aligned} \tag{62}$$

The partial differentiations indicated in Eq. (59) can be performed without difficulty. What remains to be done is to form 64 β's as in the derivation of Eqs. (49) and to develop expressions for the generalized forces $Q_1,\ldots,Q_8$. Once this has been accomplished, one can write the eight dynamical differential equations of motion (after extensive simplifying) as

$$\begin{aligned} & m(c-q_7-bc_8)[\dot{u}_6+u_1u_5-u_2u_4+(\dot{u}_1+u_2u_3)(c-q_7-bc_8)-(\dot{u}_2-u_3u_1)bs_8+2(u_1u_8-u_2u_7c_8)]+I_1\dot{u}_1-u_2u_3(I_2-I_3) \\ & =T_1+(c-q_7-bc_8)R_3 \end{aligned} \tag{63}$$

$$-mbs_8[u_6+u_1u_5-u_2u_4+(\dot{u}_1+u_2u_3)(c-q_7-bc_8)-(\dot{u}_2-u_3u_1)bs_8+2(u_1u_8-u_2u_7c_8)]+I_2\dot{u}_2-u_3u_1(I_3-I_1)=T_2-bs_8R_3 \tag{64}$$

$$\begin{aligned} & m\{bs_8[\dot{u}_5+u_3u_4-u_1u_6+\dot{u}_8+(\dot{u}_3+u_1u_2)bs_8+2u_3u_7c_8-u_1^2(c-q_7-bc_8)]-(c-q_7-bc_8)[\dot{u}_4+u_2u_6-u_3u_5+\dot{u}_7c_8 \\ & \quad -(\dot{u}_3-u_1u_2)(c-q_7-bc_8)-2u_3u_8-u_7^2s_8/b-u_2^2bs_8]\}+I_3\dot{u}_3-u_1u_2(I_1-I_2)=T_3-(c-q_7-bc_8)R_1+bs_8R_2 \end{aligned} \tag{65}$$

$$M(\dot{u}_4+u_2u_6-u_3u_5)+m[\dot{u}_4+u_2u_6-u_3u_5+\dot{u}_7c_8-(\dot{u}_3-u_1u_2)(c-q_7-bc_8)-2u_3u_8-(u_2^2+u_3^2)bs_8-u_7^2s_8/b]=R_1+S_1 \tag{66}$$

$$M(\dot{u}_5+u_3u_4-u_1u_6)+m[\dot{u}_5+u_3u_4-u_1u_6+\dot{u}_8+(\dot{u}_3+u_1u_2)bs_8+2u_3u_7c_8-(u_3^2+u_1^2)(c-q_7-bc_8)]=R_2+S_2 \tag{67}$$

$$M(\dot{u}_6+u_1u_5-u_2u_4)+m[\dot{u}_6+u_1u_5-u_2u_4+(\dot{u}_1+u_2u_3)(c-q_7-bc_8)-(\dot{u}_2-u_3u_1)bs_8+2(u_1u_8-u_2u_7c_8)]=R_3+S_3 \tag{68}$$

$$Mc_8[\dot{u}_4+u_2u_6-u_3u_5+\dot{u}_7c_8-(\dot{u}_3-u_1u_2)(c-q_7-bc_8)-2u_3u_8-(u_2^2+u_3^2)bs_8-u_7^2s_8/b]=2\tau/b-\sigma s_8+R_1c_8 \tag{69}$$

$$m[\dot{u}_5+u_3u_4-u_1u_6+\dot{u}_8+(\dot{u}_3+u_1u_2)bs_8+2u_3u_7c_8-(u_3^2+u_1^2)(c-q_7-bc_8)]=\sigma+R_2 \tag{70}$$

‡These equations are often erroneously attributed to Appell, whose paper[10] on the subject did not appear in print until 1900, twenty-one years after Gibbs' paper was published.

The amount of labor expended to arrive at these results is significantly smaller than that required to reach the corresponding point via Lagrange's equations, Hamilton's canonical equations, or the Boltzmann-Hamel equations, and it is certainly no greater than that involved in using momentum principles or D'Alembert's principle. Furthermore, the Gibbs equations are seen to furnish a procedure that is more systematic than is the application of the principles just mentioned.

As for the form of Eqs. (63-70), comparison with corresponding equations obtained by other methods, i.e., Eqs. (6-8), (10-12), (20), and (21), or Eqs. (23-25), reveals that none of the present equations is more complicated than any of its earlier counterparts, but that some are considerably simpler. For example, Eqs. (63-65) are obviously shorter than Eqs. (10-12), respectively. The reasons for this are that use of the Gibbs equations frees one from the necessity of dealing with the mass center of S, which requirement underlies the complexity of Eqs. (10-12), and it makes it possible to employ as dependent variables quantities other than generalized coordinates, such as $u_1,...,u_8$, introduced in Eqs. (44) in connection with the Boltzmann-Hamel equations.

The example at hand provides the opportunity to make one more observation ultimately of general concern. When used to simulate a motion of S, Eqs. (63-70), like Eqs. (6-8), (10-12), (20), and (21) together with $\dot{r}=x$ and $\dot{\theta}=y$, must be solved for all first derivatives, either analytically or in the course of the simulation. Now, whenever $q_8=90$ deg, the left-hand member of Eq. (69) vanishes, so that the equation no longer contains any first derivatives. The upshot of this is that the determinant of the coefficients of $\dot{u}_1,...,\dot{u}_8$, $\dot{q}_7$, and $\dot{q}_8$ vanishes, which means that one cannot solve the equations of motion when $q_8=90$ deg. This fact has nothing to do with the method used to formulate the equations, nor is it related to the introduction of the variables $u_1,...,u_8$. Indeed, it first manifests itself, albeit in less obvious form, in connection with Eqs. (20) and (21), derived by another method and not involving these variables. (The determinant of the coefficients of $\ddot{\theta}$ and $\ddot{r}$ vanishes when $\theta=90$ deg.) It is the masslessness of the linkage that is to blame. This can be seen most clearly by formulating equations that apply when mass is attributed to the linkage, which is accomplished most simply by placing a representative particle $\bar{P}$ at, say, point L_2, and assigning to this particle a mass $\bar{m}$. Specifics regarding the choice of $\bar{m}$ need not be discussed for present purposes. What is of interest is the effort that must expended to extend the analysis to account for the presence of $\bar{P}$.

Equations previously obtained by appealing to the angular momentum principle must be modified drastically to accommodate a particle at L_2, for this affects the location of the mass center S^* of S. The situation is somewhat better in connection with the equations based on D'Alembert's principle. It is a relatively simple matter to modify the first six of these, as well as the free-body diagrams in Fig. 3, to reflect the presence of an additional inertia force, but the analysis based on free-body diagrams must be carried out anew. The least laborious procedure for obtaining the desired equations is to use Eq. (59), that is, to add to the left-hand members of Eqs. (63-70) the terms $\partial(\bar{m}\bar{a}^2/2)/\partial\dot{u}_i\ (i=1,...,8)$, respectively, where $\bar{a}$ is the acceleration of $\bar{P}$ in N. In the case of Eq. (69), this term is given by

$$\begin{aligned}\partial(\bar{m}\bar{a}^2/2)/\partial\dot{u}_7 = & -\bar{m}\{c_8[\dot{u}_4+\dot{u}_7c_8-\dot{u}_3(c-bc_8)\\ & +u_2u_6-u_3u_5+u_1u_2(c-bc_8)-(u_2^2+u_3^2)bs_8]\\ & +s_8[\dot{u}_5+\dot{u}_7s_8+\dot{u}_3bs_8+u_3u_4-u_1u_6\\ & -(u_3^2+u_1^2)(c-bc_8)+u_1u_2bs_8]\}\end{aligned}\tag{71}$$

which obviously does not vanish when $q_8=\pi/2$, so that there is no longer any singularity. The remaining methods, already ruled out for other reasons, need not be discussed in this context.

While the Gibbs equations lead to equations of motion for S having a desirable form, and do so more directly than any of the other methods considered so far, there is no reason to believe that further improvements are out of the question. In particular, the right-hand sides of Eqs. (59), which require the construction of n generalized forces and n^2 coefficients β_{ij}, leave something to be desired.

IX. Last Method

The last method to be discussed applies to any material system whose configuration in a Newtonian reference frame N can be characterized by generalized coordinates $q_1,...,q_n$. The method involves two classes of quantities not employed in connection with the earlier analyses, namely *partial angular velocities* and *partial velocities,* defined as follows: If $u_1,...,u_n$, called *generalized speeds,* are introduced as linear combinations of $\dot{q}_1,...,\dot{q}_n$ by means of equations of the form

$$u_i=\sum_{j=1}^{n} W_{ij}\dot{q}_j+X_i \qquad (i=1,...,n) \tag{72}$$

where W_{ij} and X_i are functions of $q_1,...,q_n$ and the time t, and W_{ij} and $X_i\ (i,j=1,...,n)$ are chosen such that Eqs. (72) can be solved uniquely for $\dot{q}_1,...,\dot{q}_n$, then the angular velocity of any rigid body and the velocity of any point of the material system each can always be expressed uniquely as a linear function of $u_1,...,u_n$. The vector that is the coefficient of u_i in such a function is called the ith partial angular velocity of the rigid body or the ith partial velocity of the point, as the case may be. For example, with $u_1,...,u_8$ as in Eq. (44), one can express the angular velocity of A and the velocities of A^* and P as

$$\omega=u_1a_1+u_2a_2+u_3a_3 \tag{73}$$

$$v^{A^*}=u_4a_1+u_5a_2+u_6a_3 \tag{74}$$

and

$$\begin{aligned}v^P= & [u_4+u_7c_8-u_3(c-q_7-bc_8)]a_1+(u_5+u_8+u_3bs_8)a_2\\ & +[u_6+u_1(c-q_7-bc_8)-u_2bs_8]a_3\end{aligned}\tag{75}$$

respectively, and the associated partial angular velocity of A and partial velocities of A^* and P, denoted by ω_i, $v_i^{A^*}$, and $v_i^P\ (i=1,...,8)$, are thus seen to be

$$\omega_1=a_1,\quad \omega_2=a_2,\quad \omega_3=a_3,\quad \omega_4=...=\omega_8=0 \tag{76}$$

$$v_1^{A^*}=v_2^{A^*}=v_3^{A^*}=v_7^{A^*}=v_8^{A^*}=0,\ v_4^{A^*}=a_1,\quad v_5^{A^*}=a_2,\quad v_6^{A^*}=a_3 \tag{77}$$

$$\begin{gathered}v_1^P=(c-q_7-bc_8)a_3,\quad v_2^P=-bs_8a_3\\ v_3^P=-(c-q_7-bc_8)a_1+bs_8a_2\\ v_4^P=a_1,\quad v_5^P=a_2,\quad v_6^P=a_3,\quad v_7^P=c_8a_1,\quad v_8^P=a_2\end{gathered}\tag{78}$$

To obtain equations of motion with the aid of partial angular velocities and partial velocities, one begins by forming the generalized active forces K_i and generalized inertia forces $K_i^*\ (i=1,...,n)$ in accordance with the definitions

$$K_i\triangleq\sum_{j=1}^{\nu}(v_i^{P_j}\cdot F_j) \qquad (i=1,...,n) \tag{79}$$

$$K_i^*\triangleq\sum_{j=1}^{\nu}[v_i^{P_j}\cdot(-m_ja_j)] \qquad (i=1,...,n) \tag{80}$$

where ν is the number of particles in the system under consideration, $v_i^{P_j}$ is the ith partial velocity of particle P_j, F_j is the resultant of all body and contact forces acting on P_j, m_j is the mass of P_j, and a_j is the inertial acceleration of P_j. In forming K_i one can omit all nonworking forces, and, if a set of contact and/or body forces acting on a rigid body of the system is replaced with a couple of torque T together with a force S applied at a given point of the body, then the contribution of this set of forces to K_i is given by $\omega_i \cdot T + v_i \cdot S$, where ω_i is now the ith partial angular velocity of the rigid body, and v_i is the ith partial velocity of the point of application of S. For instance, for the spacecraft S,

$$K_i = v_i^P \cdot (R_1 a_1 + R_2 a_2 + R_3 a_3) + v_i^{A^*} \cdot (S_1 a_1 + S_2 a_2 + S_3 a_3)$$
$$+ \omega_i \cdot (T_1 a_1 + T_2 a_2 + T_3 a_3) + v_i^P \cdot (\sigma a_2)$$
$$+ v_i^{L_2} \cdot (-\sigma a_2) + 2\omega_i^L \cdot (\tau a_3) + 2\omega_i^A \cdot (-\tau a_3) \qquad (i=1,\ldots,8) \qquad (81)$$

Here, the first three dot products account for forces exerted on S by "external" agencies, the fourth and fifth terms represent contributions of the spring and damper that connect P to point L_2, and the last two terms deal with the torsional springs and dampers at L_1 and L_4, the symbol ω_i^L in the penultimate term being the ith partial angular velocity of both the link $L_1 - L_2$ and the link $L_3 - L_4$. Since v_i^P, $v_i^{A^*}$, and ω_i are already available in Eqs. (76-78), and $v_i^{L_2}$ and ω_i^L can be found by inspection of the velocity of L_2 and angular velocity of link $L_1 - L_2$, one easily finds by using Eq. (81) that the generalized active forces for S are

$$K_1 = T_1 + (c - q_7 - bc_8) R_3, \qquad K_2 = T_2 - bs_8 R_3$$
$$K_3 = T_3 - (c - q_7 - bc_8) R_1 + bs_8 R_2, \qquad K_4 = R_1 + S_1$$
$$K_5 = R_2 + S_2, \qquad K_6 = R_3 + S_3$$
$$K_7 = 2\tau / b - \sigma s_8 + R_1 c_8, \qquad K_8 = \sigma + R_2 \qquad (82)$$

When forming the generalized inertia forces, one can take advantage of the fact that the contribution to K_i^* of the particles of a rigid body A is given by $\omega_i \cdot T^* + v_i \cdot F^*$, where ω_i is the ith partial angular velocity of A, v_i is the ith partial velocity of the mass center of A, and T^* and F^* are, respectively, the inertia torque and inertia force introduced in conjunction with D'Alembert's principle and given in Eq. (22). For the spacecraft S, one can, therefore, write

$$K_i^* = -m v_i^P \cdot a^P + \omega_i \cdot T^* - M v_i^{A^*} \cdot a^{A^*} \qquad (i=1,\ldots,8) \qquad (83)$$

where a^P and a^{A^*} are the accelerations of P and A^* in N and

$$T^* = [u_2 u_3 (I_2 - I_3) - \dot{u}_1 I_1] a_1 + [u_3 u_1 (I_3 - I_1) - \dot{u}_2 I_2] a_2$$
$$+ [u_1 u_2 (I_1 - I_2) - \dot{u}_3 I_3] a_3 \qquad (84)$$

and thus one finds

$$K_1^* = -m(c - q_7 - bc_8)[\dot{u}_6 + u_1 u_5 - u_2 u_4$$
$$+ (\dot{u}_1 + u_2 u_3)(c - q_7 - bc_8) - (\dot{u}_2 - u_3 u_1) bs_8$$
$$+ 2(u_1 u_8 - u_2 u_7 c_8)] + u_2 u_3 (I_2 - I_3) - \dot{u}_1 I_1 \qquad (85)$$

and analogous expressions for $K_2^*,\ldots,K_8^*$.

Dynamical equations of motion are formulated by equating to zero the sum of generalized active and inertia forces:

$$K_i + K_i^* = 0 \qquad (i=1,\ldots,8) \qquad (86)$$

The equations obtained in this way for S are Eqs. (63-70). Indeed, the left-hand members of Eqs. (63-70) are precisely the negatives of the generalized inertia forces $K_1^*,\ldots,K_8^*$, and the right-hand members are the generalized active forces $K_1,\ldots,K_8$. Obviously, therefore, everything that was said about the form of the equations generated by means of the Gibbs equations applies equally well to the equations resulting from the use of Eqs. (86). But there are two clear-cut differences between these two means for reaching the same goal, the first being that $K_1,\ldots,K_8$ are formed far more easily than are their equivalents in the Gibbs equations, the eight sums

$$\sum_{j=1}^{8} \beta_{ji} Q_j \qquad (i=1,\ldots,8)$$

The labor involved in creating 64 β's is eliminated, the effort expended to find eight Q's is at least as great as that required to find eight K's, and the arduous task of performing eight summations and then simplifying the resulting expressions has no counterpart in connection with the K's. For example, consider the right-hand side of Eq. (59) for $i=4$. Since β_{14}, β_{24}, β_{34}, β_{74}, and β_{84} are equal to zero, this reduces to

$$\sum_{j=1}^{8} \beta_{j4} Q_j = \beta_{44} Q_4 + \beta_{54} Q_5 + \beta_{64} Q_6 \qquad (87)$$

where

$$Q_4 = (R_1 + S_1) c_2 c_3 - (R_2 + S_2) c_3 s_3 + (R_3 + S_3) s_2 \qquad (88)$$

$$Q_5 = (R_1 + S_1)(s_1 s_2 c_3 + s_3 c_1)$$
$$+ (R_2 + S_2)(-s_1 s_2 s_3 + c_3 c_1) - (R_3 + S_3) s_1 c_2 \qquad (89)$$

$$Q_6 = (R_1 + S_1)(-c_1 s_2 c_3 + s_3 s_1)$$
$$+ (R_2 + S_2)(c_1 s_2 s_3 + c_3 s_1) + (R_3 + S_3) c_1 c_2 \qquad (90)$$

while β_{44}, β_{54}, and β_{64} are given in Eqs. (49). Substituting, one arrives at

$$\sum_{j=1}^{8} \beta_{j4} Q_j = c_2 c_3 [(R_1 + S_1) c_2 c_3 - (R_2 + S_2) c_2 s_3 + (R_3 + S_3) s_2]$$
$$+ (s_1 s_2 c_3 + s_3 c_1)[(R_1 + S_1)(s_1 s_2 c_3 + s_3 c_1)$$
$$+ (R_2 + S_2)(-s_1 s_2 s_3 + c_3 c_1) - (R_3 + S_3) s_1 c_2]$$
$$+ (-c_1 s_2 c_3 + s_3 s_1)[(R_1 + S_1)(-c_1 s_2 c_3 + s_3 s_1)$$
$$+ (R_2 + S_2)(c_1 s_2 s_3 + c_3 s_1) + (R_3 + S_3) c_1 c_2] \qquad (91)$$

which can be simplified to

$$\sum_{j=1}^{8} \beta_{j4} Q_j = R_1 + S_1 \qquad (92)$$

Of course, since one expects to resort to a numerical solution of the equations of motion, one might forego the explicit expansion and subsequent simplifications indicated in Eqs. (91) and (92), relegating to the computer the task of forming the right-hand sides of Eqs. (59) at each step of a numerical integration. Obviously, this would lead to a very inefficient algorithm and would be a poor way to proceed, considering the fact that, by using $K_1,\ldots,K_8$, one can construct an efficient algorithm without doing any extra work.

The second difference between using the Gibbs equations and Eqs. (86) is that more work must be performed in constructing G and then evaluating the partial derivatives $\partial G / \partial u_1,\ldots,\partial G / \partial u_8$ than is needed to formulate $K_1^*,\ldots,K_8^*$ because, once the requisite kinematical analysis (which is the same for the two methods) has been carried out, the squaring of accelerations and subsequent differentiations are more laborious than the dot-multiplying of accelerations with partial velocities. For example, consider once more the problem of extending the analysis by adding a particle $\bar{P}$ of mass $\bar{m}$ at point L_2. The acceleration $\bar{a}$ of $\bar{P}$ is

$$\begin{aligned}\bar{a}=&[\dot{u}_4+\dot{u}_7c_8-\dot{u}_3(c-bc_8)-u_7^2s_8/b-2u_3u_7s_8\\&+u_2u_6-u_3u_5+u_1u_2(c-bc_8)-(u_2^2+u_3^2)bs_8]a_1\\&+[\dot{u}_5+\dot{u}_7s_8+\dot{u}_3bs_8+u_7^2c_8/b+2u_3u_7c_8+u_3u_4-u_1u_6\\&-(u_3^2+u_1^2)(c-bc_8)+u_1u_2bs_8]a_2\\&+[\dot{u}_6+\dot{u}_1(c-bc_8)-\dot{u}_2bs_8+2u_7(u_1s_8-u_2c_8)\\&+u_1u_5-u_2u_4+u_3u_1bs_8+u_2u_3(c-bc_8)]a_3\end{aligned} \quad (93)$$

To find the contribution of $\bar{P}$ to Eq. (69), one may either form $\bar{a}^2$ and then evaluate $\partial(\bar{m}\bar{a}^2/2)/\partial\dot{u}_7$, or one can dot-multiply $\bar{m}\bar{a}$ with the partial velocity $\bar{v}_7$, given by $\bar{v}_7=c_8a_1+s_8a_2$. No more needs to be said.

The generalized speeds $u_1,\ldots,u_8$, which play such a central role in connection with the method at hand, need not be introduced in the formal manner indicated by Eq. (72). Indeed, when attacking a specific problem, one should select the u's so as to arrive at especially simple expressions for angular velocities of rigid bodies and velocities of points of the system to be analyzed. This often can be done *without even introducing generalized coordinates.* For example, recognizing that the body A of the spacecraft S has six degrees of freedom, that the angular velocity of A and the velocity of A^* are important quantities, and that the unit vectors a_1, a_2, a_3, being parallel to central principal axes of inertia of A, form an especially convenient vector basis, one is well motivated to write Eqs. (73) and (74) from the outset, that is, without prior reference to Eqs. (44). Moreover, this leaves one free to describe the orientation of A in N and the position of A^* in N in terms of whatever variables (Euler parameters, direction cosines, spherical coordinates, etc.) one may wish to choose at a later stage of the analysis, rather than chaining one to a specific choice, as does the use of the Boltzmann-Hamel or Gibbs equations unless one simplifies the dynamical equations to the point at which $q_1,\ldots,q_6$ disappear. (If one employs Lagrange's or Hamilton's equations, one is again forced to select all generalized coordinates a priori, and the kind of uncoupling of kinematical and dynamical equations here being discussed is thus ruled out entirely. The momentum principles and D'Alembert's principle do not suffer from this particular failing.)

In Sec. I it was mentioned that the method considered in the present section is particularly effective when the behavior of a spacecraft can be analyzed in terms of modal coordinates. To illustrate this point, we shall formulate the equations of motion of the system shown in Fig. 5, which is a schematic representation of a spacecraft formed by a rigid body A that supports a cantilever beam B of length L and flexural rigidity EI and that moves freely in a Newtonian reference frame. For the sake of simplicity, attention will be confined to planar motions, so that the orientation of A and the position of the mass center A^* of A can be characterized by the angle ψ and the distances x_1 and x_2 shown in Fig. 5 and the behavior of B can be discussed in terms of the displacement η of a generic point Q situated a distance ξ from the fixed end.

Modal coordinates are introduced by expressing η as

$$\eta=\sum_{i=1}^{\nu}\phi_iq_i \quad (94)$$

where ϕ_i is a function of ξ, q_i is a function of t, and ν is the number of modes to be taken into account. The quantities x_1,

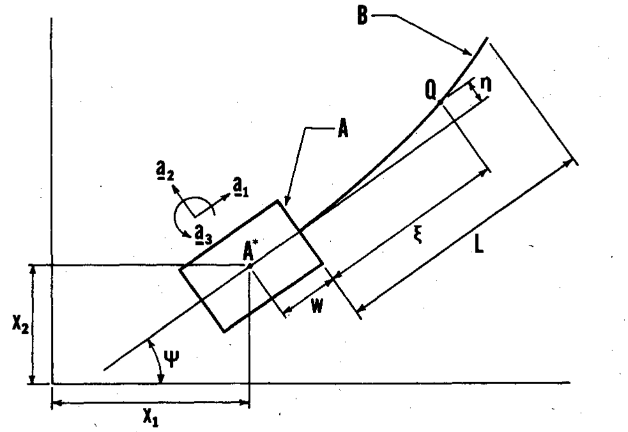

Fig. 5 Flexible spacecraft.

x_2, ψ, and $q_1,\ldots,q_\nu$ now play the roles of generalized coordinates, and associated generalized speeds $u_1,\ldots,u_{3+\nu}$ are defined as

$$u_1\triangleq\dot{x}_1\cos\psi+\dot{x}_2\sin\psi,\quad u_2\triangleq-\dot{x}_1\sin\psi+\dot{x}_2\cos\psi,\quad u_3\triangleq\dot{\psi} \quad (95)$$

and

$$u_{3+i}\triangleq\dot{q}_i \qquad (i=1,\ldots,\nu) \quad (96)$$

whereupon one can express the velocity of A^*, the angular velocity of A, the velocity of Q, and the angular velocity of a differential element of B at Q as

$$v^{A^*}=u_1a_1+u_2a_2 \qquad \omega^A=u_3a_3 \quad (97)$$

$$v^Q=\left(u_1-u_3\sum_{i=1}^{\nu}\phi_iq_i\right)a_1+\left[u_2+u_3(w+\xi)+\sum_{i=1}^{\nu}\phi_iu_{3+i}\right]a_2 \quad (98)$$

$$\omega^B=\left[\dot{\psi}+\frac{\partial}{\partial t}\left(\frac{\partial\eta}{\partial\xi}\right)\right]a_3=\left(u_3+\sum_{i=1}^{\nu}\phi_i'u_{3+i}\right)a_3 \quad (99)$$

respectively, where w is the distance from A^* to the clamped end of B and $\phi_i'\triangleq d\phi_i/d\xi$. Furthermore, partial velocities and partial angular velocities can now be formed by inspection. Letting $v_r^{A^*}$, ω_r^A, v_r^Q, ω_r^B denote, respectively, the rth partial velocity of A^*, the rth partial angular velocity of A, etc., one finds, for example,

$$v_{3+j}^{A^*}=0 \qquad \omega_{3+j}^A=0 \qquad (j=1,\ldots,\nu) \quad (100)$$

$$v_{3+j}^Q=\phi_ja_2 \qquad \omega_{3+j}^B=\phi_j'a_3 \qquad (j=1,\ldots,\nu) \quad (101)$$

Proceeding to the formulation of generalized active forces, we note that, if rotatory inertia is left out of account, the system of forces acting on a differential element of B is equivalent to a force $-(\partial V/\partial\xi)d\xi a_2$ applied to Q, and that $V=\partial M/\partial\xi=EI\partial^3\eta/\partial\xi^3$. Consequently, one can write for the generalized active force K_{3+j}

$$K_{3+j}=-\int_0^L v_{3+j}^Q\cdot[EI(\partial^4\eta/\partial\xi^4)a_2]d\xi \quad (j=1,\ldots,\nu) \quad (102)$$

or, after using Eqs. (94) and (101),

$$K_{3+j}=EI\sum_{i=1}^{\nu}q_i\int_0^L\phi_j\phi_i''''d\xi \qquad (j=1,\ldots,\nu) \quad (103)$$

which can be integrated by parts to yield

$$K_{3+j} = -EI \sum_{i=1}^{\nu} \left[q_i \int_0^L \phi_i'' \phi_j'' \mathrm{d}\xi \right] \qquad (j=1,\ldots,\nu) \tag{104}$$

where advantage has been taken of the fact that ϕ_i and ϕ_i' vanish at $\xi=0$ and ϕ_i'' and ϕ_i''' vanish at $\xi=L$. The product of EI and the integral in Eq. (104) is simply a constant, say H_{ij}. Hence

$$K_{3+j} = -\sum_{i=1}^{\nu} H_{ij} q_i \qquad (j=1,\ldots,\nu) \tag{105}$$

The acceleration of Q, required for the formulation of generalized inertia forces, is obtained by differentiating Eq. (98), which produces

$$a^Q = \left[\dot{u}_1 - u_2 u_3 - u_3^2 (w+\xi) - \sum_{i=1}^{\nu} \phi_i (\dot{u}_3 q_i + 2u_3 u_{3+i}) \right] a_1 + \left[\dot{u}_2 + u_3 u_1 + \dot{u}_3 (w+\xi) + \sum_{i=1}^{\nu} \phi_i (\dot{u}_{3+i} - u_3^2 q_i) \right] a_2 \tag{106}$$

and the generalized inertia force K^*_{3+j}, formed in accordance with Eq. (80) as

$$K^*_{3+j} = -\int_0^L v^Q_{3+j} \cdot a^Q \rho \mathrm{d}\xi \tag{107}$$

with ρ denoting the mass per unit length of B, is thus given by [see Eq. (101) for v^Q_{3+j}]

$$K^*_{3+j} = -\int_0^L \phi_j \left[\dot{u}_2 + u_3 u_1 + \dot{u}_3 (w+\xi) + \sum_{i=1}^{\nu} \phi_i (\dot{u}_{3+i} - u_3^2 q_i) \right] \rho \mathrm{d}\xi \tag{108}$$

Therefore, after defining constants E_j, F_j, and G_{ij} as

$$E_j \triangleq \int_0^L \phi_j \rho \mathrm{d}\xi \qquad F_j \triangleq \int_0^L \phi_j \xi \rho \mathrm{d}\xi \qquad G_{ij} \triangleq \int_0^L \phi_i \phi_j \rho \mathrm{d}\xi \tag{109}$$

one can write

$$K^*_{3+j} = -(\dot{u}_2 + u_3 u_1 + \dot{u}_3 w) E_j - \dot{u}_3 F_j - \sum_{i=1}^{\nu} (\dot{u}_{3+i} - u_3^2 q_i) G_{ij} \qquad (j=1,\ldots,\nu) \tag{110}$$

and, substituting into Eq. (86), one arrives directly at the equations of motion

$$\dot{u}_2 E_j + \dot{u}_3 (wE_j + F_j) + \sum_{i=1}^{\nu} \dot{u}_{3+i} G_{ij} = -u_3 u_1 E_j + \sum_{i=1}^{\nu} (u_3^2 G_{ij} - H_{ij}) q_i \qquad (j=1,\ldots,\nu) \tag{111}$$

The remaining equations of motion, that is, those obtained from $K_i + K_i^* = 0$ for $i=1,2,3$, can be formulated with equal ease. Letting m_A and m_B denote the masses of A and B (see Fig. 5), respectively, and designating as I^A the moment of inertia of A with respect to a line parallel to a_3 and passing through A^*, one is led to

$$(m_A + m_B) \dot{u}_1 - \dot{u}_3 \sum_{i=1}^{\nu} E_i q_i = (m_A + m_B) u_2 u_3 + u_3^2 m_B \left(w + \frac{L}{2} \right) + 2u_3 \sum_{i=1}^{\nu} E_i u_{3+i} \tag{112}$$

$$(m_A + m_B) \dot{u}_2 + \dot{u}_3 m_B \left(w + \frac{L}{2} \right) + \sum_{i=1}^{\nu} E_i \dot{u}_{3+i} = -(m_A + m_B) u_3 u_1 + u_3^2 \sum_{i=1}^{\nu} E_i q_i \tag{113}$$

$$\dot{u}_1 \sum_{i=1}^{\nu} E_i q_i - \dot{u}_2 m_B \left(w + \frac{L}{2} \right) - \dot{u}_3 \left[I^A + m_B \left(w^2 + wL + \frac{L^2}{3} \right) \right] - \sum_{i=1}^{\nu} (wE_i + F_i) \dot{u}_{3+i} = u_2 u_3 \sum_{i=1}^{\nu} E_i q_i + m_B u_3 u_1 \left(w + \frac{L}{2} \right) \tag{114}$$

Use of Lagrange's equations permits one to write Eqs. (111) and (114), but only after expending considerable labor in forming derivatives of the system's kinetic energy function. Moreover, the Lagrange counterparts to Eqs. (112) and (113), which are

$$(m_A + m_B) \ddot{x}_1 - \dot{u}_3 \left[m_B \left(w + \frac{L}{2} \right) \sin\psi - \cos\psi \sum_{i=1}^{\nu} E_i q_i \right] - \sin\psi \sum_{i=1}^{\nu} E_i \dot{u}_{3+i} = u_3^2 \left[m_B \left(w + \frac{L}{2} \right) \cos\psi - \lambda\theta\psi\psi \sum_{i=1}^{\nu} E_i q_i \right] + 2u_3 \cos\psi \sum_{i=1}^{\nu} E_i u_{3+i} \tag{115}$$

and

$$(m_A + m_B) \ddot{x}_2 + \dot{u}_3 \left[m_B \left(w + \frac{L}{2} \right) \cos\psi - \sin\psi \sum_{i=1}^{\nu} E_i q_i \right] + \cos\psi \sum_{i=1}^{\nu} E_i \dot{u}_{3+i} = u_3^2 \left[m_B \left(w + \frac{L}{2} \right) \cos\psi - \sin\psi \sum_{i=1}^{\nu} E_i q_i \right] + 2u_3 \sin\psi \sum_{i=1}^{\nu} E_i u_{3+i} \tag{116}$$

respectively, are significantly longer than Eqs. (112) and (113). Of course, one could use the first two of Eqs. (95) in conjunction with Eqs. (115) and (116) to recover Eqs. (112) and (113), but the algebra involved is extensive and, not having seen the latter equations, one might well overlook the possibility of bringing the equations into a more desirable form. We conclude, therefore, that Eqs. (86) are superior to Lagrange's equations for the formulation of equations governing modal coordinates.

X. Summary

It has been shown that the last method considered for the formulation of dynamical equations of complex spacecraft leads most directly to the simplest equations. It offers the analyst the greatest possible latitude in the choice of dependent variables, can be implemented systematically, and is particularly effective when modal coordinates are employed. The Gibbs equations, which lead to identical results when written in terms of the same variables, provide the basis for an equally systematic procedure, the implementation of which is, however, considerably more laborious. Use of the Boltzmann-Hamel equations becomes clearly undesirable when n, the number of degrees of freedom of a system, is greater than, say, six, because it necessitates the formulation of n^3 quantities γ_{ijk}, which consumes an inordinate amount of time and effort. Similarly, Hamilton's canonical equations and Lagrange's equations are poor tools under these circumstances, the former because they can be written explicitly only after one has inverted an $n \times n$ matrix in literal form, and the latter because they must be formulated in terms of dependent variables (generalized coordinates) which lead to unnecessarily lengthy equations, and this only after one has done much gratuitous work in differentiating energy functions. By way of contrast, D'Alembert's principle and the principles of linear and angular momentum provide viable points of departure for the derivation of equations of motion. D'Alembert's principle allows one greater flexibility than do the momentum principles because it permits one to take moments about any point whatsoever, thus facilitating the elimination of constraint forces and making it unnecessary to locate the mass center of a system. However, the fact that one can exploit this feature of the method only by making clever choices of moment centers and that such constraint forces as remain in the equations must be eliminated by algebraic means render this approach less than optimum.

When the formulation of equations of motion of a complex spacecraft is undertaken for the purpose of analyzing the stability of a particular motion or in connection with a control system design, it may occur that one is interested solely in a linearized form of the equations. The last method permits such equations to be written with a minimum of labor, for its use removes the burden of working with nonlinear expressions for anything other than velocities and angular velocities. Once these expressions are available, partial velocities and partial angular velocities can be formed by inspection. Next, one linearizes all kinematical entities in hand and employs the linear forms to construct generalized active forces and generalized inertia forces as described in Sec. IX, discarding all terms of degree two or higher that may arise in the process. Eqs. (86) then lead immediately to the desired equations.

In this review, we have confined ourselves to the consideration of holonomic systems, doing so because limitations of space prevent us from dealing in detail with the extensive literature on nonholonomic systems. (See, for example, Ref. 11.) It is worth mentioning, however, that the conclusions already drawn remain unaltered when one extends the scope of the discussion to include nonholonomic systems, and that the advantages of the last method over the rest are, if anything, more pronounced in this context. In particular, its use is to be preferred to that of Lagrange's equations, for to become applicable to nonholonomic systems these equations must be encumbered with Lagrange multipliers or, equivalently, unknowns representing constraint forces, quantities whose subsequent elimination is simply an additional burden. Details regarding the formulation of partial velocities and partial angular velocities for nonholomic systems and, with their aid, equations of the form of Eq. (86) are set forth in Ref. 12.

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